

REAL HYPERSURFACES EQUIPPED WITH ξ -PARALLEL STRUCTURE JACOBI OPERATOR IN $\mathbb{C}P^2$ OR $\mathbb{C}H^2$

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ABSTRACT. The ξ -parallelness condition of the structure Jacobi operator of real hypersurfaces has been studied in combination with additional conditions. In the present paper we study three dimensional real hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ equipped with ξ -parallel structure Jacobi operator. We prove that they are Hopf hypersurfaces and if additional $\eta(A\xi) \neq 0$, we give the classification of them.

Keywords: Real hypersurface, ξ -parallel structure Jacobi operator, Complex projective space, Complex hyperbolic space.

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1 Introduction

A complex n -dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n$, a complex Euclidean space \mathbb{C}^n or a complex hyperbolic space $\mathbb{C}H^n$ if $c > 0$, $c = 0$ or $c < 0$ respectively.

The study of real hypersurfaces in a nonflat complex space form is a classical problem in Differential Geometry. Let M be a real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (φ, ξ, η, g) . The structure vector field ξ is called principal if $A\xi = \alpha\xi$ holds on M , where A is the shape operator of M in $M_n(c)$ and α is a smooth function. A real hypersurface is called *Hopf hypersurface* if ξ is principal.

Takagi in [14] classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ and Berndt in [1] classified Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$. Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. Then we state the following theorems due to Okumura [11] for $\mathbb{C}P^n$ and Montiel and Romero [9] for $\mathbb{C}H^n$ respectively.

Theorem 1.1 *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$, $c \neq 0$. If it satisfies $A\varphi - \varphi A = 0$, then M is locally congruent to one of the following hypersurfaces:*

- In case $\mathbb{C}P^n$
(A_1) a geodesic hypersphere of radius r , where $0 < r < \frac{\pi}{2}$,

(A₂) a tube of radius r over a totally geodesic $\mathbb{C}P^k$, ($1 \leq k \leq n-2$), where $0 < r < \frac{\pi}{2}$.

- In case $\mathbb{C}H^n$

(A₀) a horosphere in $\mathbb{C}H^n$, i.e a Montiel tube,

(A₁) a geodesic hypersphere or a tube over a hyperplane $\mathbb{C}H^{n-1}$,

(A₂) a tube over a totally geodesic $\mathbb{C}H^k$ ($1 \leq k \leq n-2$).

Since 2006 many authors have studied real hypersurfaces whose structure Jacobi operator is parallel ($\nabla l = 0$). Ortega, Perez and Santos [12] proved the nonexistence of real hypersurfaces in non-flat complex space form with parallel structure Jacobi operator $\nabla l = 0$. Perez, Santos and Suh [13] continuing the work of [12] considered a weaker condition (\mathbb{D} -parallelness), that is $\nabla_X l = 0$ for any vector field X orthogonal to ξ . They proved the non-existence of such real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$.

Kim and Ki in [7] classified real hypersurfaces if $\nabla_\xi l = 0$ and $S\varphi = \varphi S$. Ki and Liu [5] proved that real hypersurfaces satisfying $\nabla_\xi l = 0$ and $lS = Sl$ are Hopf hypersurfaces provided that the scalar curvature is non-negative. Ki, et.al. in [6] classified real hypersurfaces satisfying $\nabla_\xi l = 0$ and $\nabla_\xi S = 0$. Kim et.al. in [8] studied the real hypersurfaces satisfying $g(\nabla_\xi \xi, \nabla_\xi \xi) = \mu^2 = \text{const}$, $6\mu^2 + \frac{c}{4} \neq 0$ and classified those whose l is ξ -parallel. Cho and Ki [3] classified real hypersurfaces satisfying $Al = lA$ and $\nabla_\xi l = 0$.

Recently Ivey and Ryan, in [4] studied real hypersurfaces in $M_2(c)$.

Motivated by all the above conclusions we study real hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ equipped with ξ -parallel structure Jacobi operator, i.e. $\nabla_\xi l = 0$. More precisely, the following relation holds:

$$(\nabla_\xi l)X = 0. \quad (1.1)$$

We prove the following theorem

Main Theorem: *Let M be a connected real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with ξ -parallel structure Jacobi operator. Then M is a Hopf hypersurface. Further, if $\eta(A\xi) \neq 0$, then :*

- in the case of $\mathbb{C}P^2$, M is locally congruent to a geodesic sphere, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$,
- in the case of $\mathbb{C}H^2$, M is locally congruent to a horosphere,
or to a geodesic sphere
or to a tube over the hyperplane $\mathbb{C}H^1$.

2 Preliminaries

Throughout this paper all manifolds, vector fields e.t.c. are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces are supposed to be oriented and without boundary.

Let M be a real hypersurface immersed in a nonflat complex space form $(M_n(c), G)$ with almost complex structure J of constant holomorphic sectional curvature c . Let N be a unit normal vector field on M and $\xi = -JN$. For a vector field X tangent to M we can write $JX = \varphi(X) + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connection $\bar{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M :

$$\bar{\nabla}_Y X = \nabla_Y X + g(AY, X)N$$

$$\bar{\nabla}_X N = -AX$$

where g is the Riemannian metric on M induced from G of $M_n(c)$ and A is the shape operator of M in $M_n(c)$. M has an almost contact metric structure (φ, ξ, η) induced from J on $M_n(c)$ where φ is a (1,1) tensor field and η a 1-form on M such that (see [2])

$$g(\varphi X, Y) = G(JX, Y), \quad \eta(X) = g(X, \xi) = G(JX, N).$$

Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1 \quad (2.1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y) \quad (2.2)$$

$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi \quad (2.3)$$

Since the ambient space is of constant holomorphic sectional curvature c , the equations of Gauss and Codazzi for any vector fields X, Y, Z on M are respectively given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \quad (2.4)$$

$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi] \quad (2.5)$$

where R denotes the Riemannian curvature tensor on M .

For every point $P \in M$, the tangent space $T_P M$ can be decomposed as following:

$$T_P M = \text{span}\{\xi\} \oplus \ker \eta$$

where $\ker(\eta) = \{X \in T_P M : \eta(X) = 0\}$. Due to the above decomposition, the vector field $A\xi$ is decomposed as follows:

$$A\xi = \alpha\xi + \beta U \quad (2.6)$$

where $\beta = |\varphi \nabla_\xi \xi|$ and $U = -\frac{1}{\beta} \varphi \nabla_\xi \xi \in \ker(\eta)$, provided that $\beta \neq 0$.

3 Auxiliary relations

Let M be a real hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, i.e. $M_2(c)$, $c \neq 0$. We consider the open subset \mathcal{N} of M such that:

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\}.$$

Furthermore, we consider \mathcal{V}, Ω open subsets of \mathcal{N} such that:

$$\mathcal{V} = \{P \in \mathcal{N} : \alpha = 0, \text{ in a neighborhood of } P\},$$

$$\Omega = \{P \in \mathcal{N} : \alpha \neq 0, \text{ in a neighborhood of } P\},$$

where $\mathcal{V} \cup \Omega$ is open and dense in the closure of \mathcal{N} .

Lemma 3.1 *Let M be a real hypersurface in $M_2(c)$, equipped with ξ -parallel structure Jacobi operator. Then \mathcal{V} is empty.*

Proof: Let $\{U, \varphi U, \xi\}$ be a local orthonormal basis on \mathcal{V} . The relation (2.6) takes the form $A\xi = \beta U$. The first relation of (2.3) for $X = \xi$, taking into account the latter implies

$$\nabla_\xi \xi = \beta \varphi U.$$

Relation (1.1) for $X = \xi$, because of the above relation yields:

$$\nabla_\xi(l\xi) = l\nabla_\xi \xi \Rightarrow \beta \varphi U = 0,$$

which leads to a contradiction and this completes the proof of Lemma 3.1. \square

In what follows we work on Ω , where $\alpha \neq 0$ and $\beta \neq 0$.

Lemma 3.2 *Let M be a real hypersurface in $M_2(c)$, equipped with ξ -parallel structure Jacobi operator. Then the following relations hold in Ω :*

$$AU = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right)U + \beta\xi, \quad A\varphi U = -\frac{c}{4\alpha}\varphi U \quad (3.1)$$

$$\nabla_\xi \xi = \beta \varphi U, \quad \nabla_U \xi = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha} \right) \varphi U, \quad \nabla_{\varphi U} \xi = \frac{c}{4\alpha} U \quad (3.2)$$

$$\nabla_\xi U = \kappa_1 \varphi U, \quad \nabla_U U = \kappa_2 \varphi U, \quad \nabla_{\varphi U} U = \kappa_3 \varphi U - \frac{c}{4\alpha} \xi \quad (3.3)$$

$$\nabla_\xi \varphi U = -\kappa_1 U - \beta \xi, \quad \nabla_U \varphi U = -\kappa_2 U - \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha} \right) \xi, \quad \nabla_{\varphi U} \varphi U = -\kappa_3 U \quad (3.4)$$

$$\kappa \kappa_1 = 0, \quad (\xi \kappa) = 0, \quad (3.5)$$

where $\kappa, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M .

Proof: Let $\{U, \varphi U, \xi\}$ be a local orthonormal basis of Ω .

The first relation of (2.3) for $X = \xi$ implies: $\nabla_\xi \xi = \beta \varphi U$ and so relation (1.1) for $X = \xi$, taking into account the latter, gives:

$$l\varphi U = 0. \quad (3.6)$$

Relation (2.4) for $X = \varphi U$ and $Y = Z = \xi$ gives: $l\varphi U = \frac{c}{4}\varphi U + \alpha A\varphi U$, which because of (3.6) implies the second of (3.1). Relation (2.4) for $X = U$ and $Y = Z = \xi$, we have:

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi \quad (3.7)$$

The scalar products of (3.7) with φU and U , because of (2.6) and the second of (3.1) imply the first of (3.1), where $\kappa = g(lU, U)$.

The first relation of (2.3), for $X = U$ and $X = \varphi U$. taking into consideration relations (3.1), gives the rest of relation (3.2).

From the well known relation: $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for $X, Y, Z \in \{\xi, U, \varphi U\}$ we obtain (3.3) and (3.4), where $\kappa_1, \kappa_2, \kappa_3$ are smooth functions in Ω .

On the other hand

$$\begin{aligned} \xi \kappa &= \xi g(lU, U) \\ \Rightarrow \xi \kappa &= g(\nabla_\xi (lU), U) + g(lU, \nabla_\xi U) \\ \Rightarrow \xi \kappa &= g((\nabla_\xi l)U + l(\nabla_\xi U), U) + g(lU, \nabla_\xi U) \\ \Rightarrow \xi \kappa &= g(l(\nabla_\xi U), U) + g(lU, \nabla_\xi U) \end{aligned}$$

The above relation because of (3.3), (3.6) and (3.7) yields:

$$\xi \kappa = g(\kappa_1 l\varphi U, U) + g(lU, \kappa_1 \varphi U) \Rightarrow \xi \kappa = 0$$

On the other hand:

$$\begin{aligned}
\xi g(l\varphi U, U) &= 0 \\
\Rightarrow g(\nabla_\xi(l\varphi U), U) + g(l\varphi U, \nabla_\xi U) &= 0 \\
\Rightarrow g((\nabla_\xi l)\varphi U + l(\nabla_\xi \varphi U), U) + g(l\varphi U, \nabla_\xi U) &= 0
\end{aligned}$$

From the above equation because of (1.1), (2.6), (3.4), (3.6) and $\kappa = g(lU, U)$ we obtain:

$$g(l(-\kappa_1 U - \beta \xi), U) = 0 \Rightarrow \kappa \kappa_1 = 0$$

□

Relation (2.5) for $X \in \{U, \varphi U\}$ and $Y = \xi$, because of Lemma 3.2 yields:

$$U\beta = \xi\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right) \quad (3.8)$$

$$U\alpha = \xi\beta \quad (3.9)$$

$$\frac{\beta^2 \kappa_1}{\alpha} = \kappa + \beta \kappa_2 + \frac{c}{4\alpha}\left(\frac{\kappa}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) \quad (3.10)$$

$$(\varphi U)\beta = \frac{\kappa_1 \beta^2}{\alpha} + \beta^2 + \frac{c}{4\alpha}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right) \quad (3.11)$$

$$\xi\alpha = \frac{4\alpha^2 \kappa_3 \beta}{c} \quad (3.12)$$

$$(\varphi U)\alpha = \beta\left(\kappa_1 + \alpha + \frac{3c}{4\alpha}\right) \quad (3.13)$$

Furthermore, relation (2.5), for $X = U$ and $Y = \varphi U$, due to Lemma 3.2 and (3.10), implies:

$$(\varphi U)\kappa = -\frac{c\beta\kappa_1}{4\alpha} + \kappa\beta + \kappa\kappa_2 - c\beta \quad (3.14)$$

$$U\alpha = \frac{4\kappa_3\alpha}{c}(\beta^2 + \kappa) \quad (3.15)$$

Using the relations (3.9)-(3.15) and Lemma 3.2 we obtain:

$$\begin{aligned}
[U, \xi]\left(\frac{c}{4\alpha}\right) &= (\nabla_U \xi - \nabla_\xi U)\frac{c}{4\alpha} \\
\Rightarrow [U, \xi]\left(\frac{c}{4\alpha}\right) &= -\frac{c\beta}{4\alpha^2}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha} - \kappa_1\right)(\kappa_1 + \alpha + \frac{3c}{4\alpha}) \quad (3.16)
\end{aligned}$$

$$\begin{aligned}
[U, \xi]\left(\frac{c}{4\alpha}\right) &= (U(\xi\frac{c}{4\alpha})) - (\xi(U\frac{c}{4\alpha})) \\
\Rightarrow [U, \xi]\left(\frac{c}{4\alpha}\right) &= -\beta\kappa_3^2 - \beta(U\kappa_3) + \left(\frac{\beta^2}{\alpha} + \frac{\kappa}{\alpha}\right)(\xi\kappa_3) \quad (3.17)
\end{aligned}$$

Similarly:

$$[U, \varphi U]\left(\frac{c}{4\alpha}\right) = \kappa_2 \kappa_3 \left(\frac{\beta^2}{\alpha} + \frac{\kappa}{\alpha}\right) + \beta \kappa_3 \left(\frac{\kappa}{\alpha} - \frac{c}{2\alpha} + \frac{\beta^2}{\alpha}\right) + \frac{c\beta\kappa_3}{4\alpha^2}(\kappa_1 + \alpha + \frac{3c}{4\alpha}) \quad (3.18)$$

$$[U, \varphi U]\left(\frac{c}{4\alpha}\right) = \frac{2\kappa_3\beta^3\kappa_1}{\alpha^2} + \frac{\kappa_3\beta^3}{\alpha} + \frac{5c\kappa_3\beta}{4\alpha^3}(\beta^2 + \kappa) - \frac{c\beta\kappa_1\kappa_3}{4\alpha^2} - \frac{c\beta\kappa_3}{4\alpha} - \frac{5c^2\beta\kappa_3}{16\alpha^3} - \frac{c\beta}{4\alpha^2}(U\kappa_1) - \frac{\beta\kappa\kappa_3}{\alpha} + \frac{\kappa_3}{\alpha}((\varphi U)\kappa) + \left(\frac{\beta^2}{\alpha} + \frac{\kappa}{\alpha}\right)((\varphi U)\kappa_3) \quad (3.19)$$

$$[\varphi U, \xi]\left(\frac{c}{4\alpha}\right) = -\kappa_3\left(\kappa_1 + \frac{c}{4\alpha}\right)\left(\frac{\beta^2}{\alpha} + \frac{\kappa}{\alpha}\right) - \beta^2\kappa_3 \quad (3.20)$$

$$[\varphi U, \xi]\left(\frac{c}{4\alpha}\right) = -\frac{2\kappa_1\kappa_3\beta^2}{\alpha} - \kappa_3\beta^2 - \frac{7c\beta^2\kappa_3}{4\alpha^2} + \frac{c^2\kappa_3}{16\alpha^2} + \frac{c\kappa\kappa_3}{2\alpha^2} - \beta(\varphi U)\kappa_3 + \kappa\kappa_3 + \frac{c\beta}{4\alpha^2}(\xi\kappa_1). \quad (3.21)$$

Due to the first relation of (3.5), we consider Ω_1 the open subset of Ω such that:

$$\Omega_1 = \{P \in \Omega : \kappa_1 \neq 0, \text{ in a neighborhood of } P\}.$$

So in Ω_1 , we have: $\kappa = 0$.

In Ω_1 relation (3.14), since $\kappa = 0$, yields:

$$\kappa_1 = -4\alpha \quad (3.22)$$

and from relation (3.10), taking into account (3.22), we get:

$$\kappa_2 = -4\beta - \frac{c\beta}{4\alpha^2} + \frac{c^2}{16\alpha^2\beta} \quad (3.23)$$

From (3.20) and (3.21), using (3.12), (3.22) and (3.23) we obtain:

$$\beta(\varphi U)\kappa_3 = -\frac{3c\beta^2\kappa_3}{2\alpha^2} + \frac{c^2\kappa_3}{16\alpha^2} \quad (3.24)$$

From (3.18), (3.19), using (3.15), (3.22), (3.23) and (3.24), we obtain:

$$\kappa_3(4\alpha^2 - c) = 0. \quad (3.25)$$

Because of (3.25), let Ω'_1 be the open subset of Ω_1 such that:

$$\Omega'_1 = \{P \in \Omega_1 : \kappa_3 \neq 0, \text{ in a neighborhood of } P\}.$$

So in Ω'_1 we obtain: $c = 4\alpha^2$. Differentiation of the latter with respect to ξ , implies $\xi\alpha = 0$ which because of (3.12) leads to $\kappa_3 = 0$, which is impossible. So Ω'_1 is empty and $\kappa_3 = 0$ in Ω_1 .

Lemma 3.3 *Let M be a real hypersurface in $M_2(c)$, equipped with ξ -parallel structure Jacobi operator. Then Ω_1 is empty.*

Proof: We resume that in Ω_1 we have:

$$\kappa = \kappa_3 = 0 \quad (3.26)$$

and relations (3.22), (3.23) and (3.24) hold.

Relations (3.8), (3.9), (3.12) and (3.15), because of (3.5) and (3.26), yield:

$$U\alpha = U\beta = \xi\alpha = \xi\beta = 0 \quad (3.27)$$

In Ω_1 , combining (3.16) and (3.17) and taking into account (3.22) and (3.26), we obtain:

$$\left(\frac{c}{4\alpha} - \alpha\right)\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha\right) = 0 \quad (3.28)$$

Owing to (3.28), let Ω_{11} be the open subset of Ω_1 , such that:

$$\Omega_{11} = \{P \in \Omega_1 : c \neq 4\alpha^2, \text{ in a neighborhood of } P\}.$$

From (3.28) in Ω_{11} , we have: $4\alpha = -\frac{\beta^2}{\alpha} + \frac{c}{4\alpha}$. Differentiation of the latter along φU , because of (3.11), (3.13), (3.22), (3.26) and the last relation yields $c = 0$, which is impossible. Hence, Ω_{11} is empty.

So in Ω_1 the relation $c = 4\alpha^2$ holds. Due to the last relation and (3.22), the relation (3.11) becomes:

$$(\varphi U)\beta = -(\alpha^2 + 2\beta^2). \quad (3.29)$$

From (3.27) we have $[U, \xi]\beta = U(\xi\beta) - \xi(U\beta) \Rightarrow [U, \xi]\beta = 0$. On the other hand, from (3.2) and (3.3) we obtain $[U, \xi]\beta = (\nabla_U \xi - \nabla_\xi U)\beta \Rightarrow [U, \xi]\beta = \frac{1}{\alpha}(3\alpha^2 + \beta^2)(\varphi U)\beta$. The last two relations imply $(\varphi U)\beta = 0$. Therefore, from (3.29) we obtain $\alpha^2 + 2\beta^2 = 0$, which is a contradiction. Hence, Ω_1 is empty. \square

Since Ω_1 is empty, in Ω we have $\kappa_1 = 0$. So from relations (3.20) and (3.21) we obtain:

$$\beta(\varphi U)\kappa_3 = \frac{\kappa_3}{16\alpha^2}[c^2 - 24c\beta^2 + 12c\kappa + 16\alpha^2\kappa].$$

Furthermore, the combination of relations (3.18) and (3.19), using (3.10) and (3.14), implies:

$$(\beta^2 + \kappa)(\varphi U)\kappa_3 = \frac{c\beta\kappa_3}{16\alpha^2}[16\alpha^2 - 24(\beta^2 + \kappa) + 9c].$$

From the last two relations we obtain:

$$\kappa_3[c^2\kappa + 12c\kappa^2 + 12c\beta^2\kappa + 16\alpha^2\beta^2\kappa - 16c\alpha^2\beta^2 + 16\alpha^2\kappa^2 - 8c^2\beta^2] = 0$$

Due to the above relation, we consider Ω_2 the open subset of Ω , such that:

$$\Omega_2 = \{P \in \Omega : \kappa_3 \neq 0, \text{ in a neighborhood of } P\},$$

so in Ω_2 the following relation holds:

$$c^2\kappa + 12c\kappa^2 + 12c\beta^2\kappa + 16\alpha^2\beta^2\kappa - 16c\alpha^2\beta^2 + 16\alpha^2\kappa^2 - 8c^2\beta^2 = 0. \quad (3.30)$$

Differentiating (3.30) with respect to ξ and using (3.5), (3.9), (3.12) and (3.15) we obtain:

$$\begin{aligned} 8\alpha^2\beta^2\kappa - 8c\alpha^2\beta^2 + 8\alpha^2\kappa^2 + 3c\beta^2\kappa - 2c^2\beta^2 + 3c\kappa^2 - 4c\alpha^2\kappa \\ - 2c^2\kappa = 0 \end{aligned} \quad (3.31)$$

From (3.30) and (3.31) we obtain:

$$5c\kappa + 6\kappa^2 + 6\beta^2\kappa - 4c\beta^2 + 8\alpha^2\kappa = 0. \quad (3.32)$$

Differentiating (3.32) with respect to ξ and using (3.5), (3.9), (3.12) and (3.15) we have: $4\kappa\alpha^2 = (2c - 3\kappa)(\beta^2 + \kappa)$. The last relation with (3.32) imply: $\kappa = 0$. Substituting the latter in (3.30) gives $c = -2\alpha^2$. Differentiation of the last relation with respect to φU and taking into account (3.13), $c = -2\alpha^2$ and $\kappa_1 = 0$ results in $\alpha = 0$, which is impossible.

So Ω_2 is empty and in Ω we get: $\kappa_3 = 0$.

Lemma 3.4 *Let M be a real hypersurface in $M_2(c)$, equipped with ξ -parallel structure Jacobi operator. Then Ω is empty.*

Proof: We resume that in Ω the following relation holds:

$$\kappa_1 = \kappa_3 = 0. \quad (3.33)$$

Relations (3.8), (3.9), (3.12), (3.15), because of (3.5) and (3.33) yield:

$$U\alpha = U\beta = \xi\alpha = \xi\beta = 0 \quad (3.34)$$

In Ω the combination of (3.16), (3.17) and taking into account (3.33), implies:

$$(4\alpha^2 + 3c)(\beta^2 + \kappa - \frac{c}{4}) = 0 \quad (3.35)$$

Due to (3.35), we consider Ω_3 the open subset of Ω such that:

$$\Omega_3 = \{P \in \Omega : \beta^2 + \kappa \neq \frac{c}{4}, \text{ in a neighborhood of } P\}.$$

So in Ω_3 the following relation holds:

$$c = -\frac{4\alpha^2}{3}. \quad (3.36)$$

Differentiation of (3.36) with respect to φU implies:

$$(\varphi U)\alpha = 0. \quad (3.37)$$

Because of (3.34) we have $[U, \xi]\beta = U(\xi\beta) - \xi(U\beta) \Rightarrow [U, \xi]\beta = 0$. On the other hand due to (3.2), (3.3) and (3.33) we get $[U, \xi]\beta = (\nabla_U \xi - \nabla_\xi U)\beta \Rightarrow [U, \xi]\beta = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + \frac{\kappa}{\alpha})(\varphi U)\beta$. Combination of the last relations imply:

$$(\varphi U)\beta = 0 \quad (3.38)$$

From (3.11), owing to (3.33), (3.36) and (3.38) yields $2\beta^2 = \kappa + \frac{\alpha^2}{3}$. Differentiation of the last relation with respect to φU and taking into account (3.37) and (3.38) imply $(\varphi U)\kappa = 0$. So from (3.14), because of the latter and (3.33), we obtain $\kappa(\beta + \kappa_2) = c\beta$. The combination of the latter with (3.10) and taking into account (3.33), (3.36) and $2\beta^2 = \kappa + \frac{\alpha^2}{3}$ imply:

$$\alpha^2 = 18\beta^2 \quad \kappa_2 = 5\beta \quad \kappa = -4\beta^2 \quad (3.39)$$

The relations of Lemma 3.2 in Ω_3 , because of (3.36) and (3.39) become:

$$AU = \frac{\alpha}{6}U + \beta\xi, \quad A\varphi U = \frac{\alpha}{3}\varphi U \quad (3.40)$$

$$\nabla_\xi \xi = \beta\varphi U, \quad \nabla_U \xi = \frac{\alpha}{6}\varphi U, \quad \nabla_{\varphi U} \xi = -\frac{\alpha}{3}U, \quad (3.41)$$

$$\nabla_\xi U = 0, \quad \nabla_U U = 5\beta\varphi U, \quad \nabla_{\varphi U} U = \frac{\alpha}{3}\xi, \quad (3.42)$$

$$\nabla_\xi \varphi U = -\beta\xi, \quad \nabla_U \varphi U = -5\beta U - \frac{\alpha}{6}\xi, \quad \nabla_{\varphi U} \varphi U = 0. \quad (3.43)$$

The relation (2.4), because of (3.36), (3.39) and (3.40) implies: $R(U, \varphi U)U = 23\beta^2\varphi U$. On the other hand $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$, because of (3.34), (3.36), (3.39) and (3.41)-(3.43) yields: $R(U, \varphi U)U = 26\beta^2\varphi U$. The combination the last two relations implies $\beta = 0$, which is impossible in Ω_3 .

So Ω_3 is empty and in Ω the following relation holds

$$\beta^2 + \kappa = \frac{c}{4}. \quad (3.44)$$

In Ω (3.10) becomes:

$$\kappa + \beta\kappa_2 = 0. \quad (3.45)$$

Differentiating (3.44) with respect to φU and using (3.11), (3.14), (3.33), (3.44) and (3.45) we obtain: $\beta^2 = -\frac{c}{4}$. Differentiation of the last relation along φU implies $(\varphi U)\beta = 0$, which because of (3.11), (3.33) and (3.44) yields $\beta = 0$, which is a contradiction. Therefore, Ω is empty and this completes the proof of Lemma 3.4. \square

From Lemmas 3.1 and 3.4, we conclude that \mathcal{N} is empty and we lead to the following result:

Proposition 3.5 *Every real hypersurface in $M_2(c)$, equipped with ξ -parallel structure Jacobi operator, is a Hopf hypersurface.*

4 Proof of Main Theorem

Since M is a Hopf hypersurface, due to Theorem 2.1 [10] we have that α is a constant. We suppose that $\alpha \neq 0$. We consider a unit vector field $e \in \mathbb{D}$, such that $Ae = \lambda e$, then $A\varphi e = \nu\varphi e$ at some point $P \in M$, where $\{e, \varphi e, \xi\}$ is a local orthonormal basis. Then the following relation holds on M , (Corollary 2.3 [10]):

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \quad (4.1)$$

The first relation of (2.3) for $X = e$ implies:

$$\nabla_e \xi = \lambda\varphi e. \quad (4.2)$$

Relation (2.4) for $X = e$ and $Y = Z = \xi$ yields:

$$le = \frac{c}{4}e + \alpha Ae. \quad (4.3)$$

From relation (1.1) for $X = e$, we obtain:

$$\nabla_\xi(le) = l\nabla_\xi e. \quad (4.4)$$

From (2.4) for $X = \nabla_\xi e$ and $Y = Z = \xi$, we get:

$$l\nabla_\xi e = \frac{c}{4}\nabla_\xi e + \alpha A(\nabla_\xi e). \quad (4.5)$$

Substitution in (4.4) of (4.3) and (4.5) yields:

$$(\nabla_\xi A)e = 0. \quad (4.6)$$

The relation (2.5) for $X = \xi$ and $Y = e$, taking into account (4.6), we get:

$$(\nabla_e A)\xi = -\frac{c}{4}\varphi e \quad (4.7)$$

Finally, the scalar product of (4.7) with φe , taking into consideration (4.1), (4.2) and $A\varphi e = \nu\varphi e$ yields:

$$\begin{aligned} g(\nabla_e(A\xi) - A\nabla_e\xi, \varphi e) &= -\frac{c}{4} \\ \Rightarrow \alpha\lambda &= -\frac{c}{4} + \lambda\nu \Rightarrow \lambda = \nu. \end{aligned}$$

Then $Ae = \lambda e$ and $A\varphi e = \lambda\varphi e$, therefore we obtain:

$$(A\varphi - \varphi A)X = 0, \quad \forall X \in TM.$$

From the above relation Theorem 1.1 holds. Since $\alpha \neq 0$ we can not have the geodesic sphere of radius $r = \frac{\pi}{4}$ and this completes the Proof of Main Theorem.

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